Degradation Models and Crack Propagation

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Abstract

In experiments where failure times are sparse, degradation analysis is useful for the analysis of failure time distributions in reliability studies. This research investigates the link between a practitioner’s selected degradation model and the resulting lifetime model. Simple additive and multiplicative models with single random effects are featured. Results show that seemingly innocuous assumptions of the degradation path create surprising restrictions on the lifetime distribution. These constraints are described in terms of failure rate and distribution classes.

Keywords: additive model, bathtub and increasing failure rates, crack growth, random effects, stochastic ordering

1. Introduction

Reliability testing based on time-to-failure measurements are often hampered by the lack of observed failures. Accelerated life testing (ALT) can hasten product failure during test intervals by stressing the product beyond its normal use. In many tests, the failure data is supplemented by degradation data, which refers to measurements of product wear available at one or more time points in the reliability test. The product lifetime is defined to be the first point in time when the degradation reaches a prespecified threshold.

Recently, degradation data has become a necessity due to extremely high product reliability that yields sparse failure data in life tests. Meeker and Escobar [4] offer a comprehensive guide to degradation analysis for various life tests, including ALT, and show that degradation analysis has great potential to improve upon reliability analysis.

However, degradation analysis can also introduce the potential for inconsistency in the experimenter’s treatment of the data. The key to the analysis is the perceived link between the degradation measurements and the failure time. The degradation model actually implies a lifetime distribution, but those distributions rarely conform to industry needs for lifetime data analysis. Typically, the resulting estimate of the lifetime distribution must be solved numerically with estimate uncertainty computed using simulation and intensive re-sampling methods such as bootstrap procedures.

In this article, we investigate the link between a chosen degradation model and the resulting lifetime model. We consider simple additive and multiplicative models and seek degradation models that lead to particular families (e.g., increasing failure rate or “bathtub” shaped failure rate) of lifetime distributions.

2. Degradation Models

Degradation models vary markedly across the fields of reliability modeling. Many practical problems can be modelled with a linear (or log-linear) rate of degradation, such as Lu, Park and Yang’s ([3]) random effects model for semiconductor degradation. Bogdanoff and Kozin [2] employ both linear and more complex nonlinear models to characterize degradation in materials testing (e.g., crack growth). The random effects in the degradation model are the key link between the degradation function and the resulting lifetime distribution. Some degradation models employ a single random effect as an error term in an additive model. Contemporary degradation models are apt to consider several random effects that enter into the degradation function in nonlinear form, including multiplicative terms. For this research, basic additive and multiplicative models are considered, and the focus is on a single random effect. For simplicity, we assume the item-to-item variation (characterized by this random effect) dominates the variability in degradation, and we ignore error terms, such as within-item variability.

2.1 Additive Degradation Model

Consider the general additive degradation model

\[ D(t; X, \Theta) = \eta(t; \Theta) + X, \]  

where \( \eta(t; \Theta) \) is a deterministic mean degradation path with fixed effect parameters \( \Theta \) for time \( t \geq 0 \). We focus on \( \eta \) being monotonic since most degradation measurements have this quality. Bae and Kvam [1] consider the non-monotonic degradation
of light displays, but this example is the exception to the norm. \( X \) represents random variation around a mean degradation level \( \eta(t; \Theta) \) with a cumulative distribution function (cdf) \( G_X \) and a probability density function (pdf) \( g_X \).

Let \( F_{AD}(t) \) denote the lifetime distribution generated by the DDP in the additive degradation model (1). Then
\[
F_{AD}(t) = Pr[\{t; X, \Theta) < \Df\} = G_X(\Df - \eta(t; \Theta)),
\]
with survival function \( F_{AD}(t) = 1 - F_{AD}(t) = 1 - G_X(\Df - \eta(t; \Theta)) \). For the IDP, the lifetime distribution is \( F_{AI}(t) = 1 - G_X(\Df - \eta(t; \Theta)) \). This defines the transformation to lifetime random variable \( T = T(X) \).

Note that (2) is a valid distribution only if \( G_X(\Df - \eta(0; \Theta)) = 0 \) and \( G_X(\Df - \eta(\infty; \Theta)) = 1 \). For the IDP, we require that \( G_X(\Df - \eta(0; \Theta)) = 1 \) and \( G_X(\Df - \eta(\infty; \Theta)) = 0 \). If \( \eta(t; \Theta) \) has finite asymptotes, for example, the additive degradation model will not necessarily produce a proper lifetime distribution function. Along with these constraints on \( G \), we assume \( G \) (and hence \( F \)) is twice differentiable on \( (0, \infty) \). Hereafter, we write \( \eta(t) \) for \( \eta(t; \Theta) \) for simplicity. Though this simple relationship between \( T \) and \( X \) is plain to the reader, its effect on the lifetime properties is more hidden. This can be shown through the failure rate function.

The failure rate corresponding to the DDP, \( r_{AD}(t) \), is defined by
\[
r_{AD}(t) = \frac{f_{AD}(t)}{1 - F_{AD}(t)} = \frac{[G_X(\Df - \eta(t))]}{1 - G_X(\Df - \eta(t))} \]
\[
= \frac{-\eta'(t) \cdot g_X(\Df - \eta(t))}{1 - G_X(\Df - \eta(t))} \]
\[
= \frac{-\eta'(t) \cdot r_X(\Df - \eta(t))},
\]
where \( r_X \) denotes the failure rate of the degradation random variable \( X \). If \( \eta(t) \) is continuous and decreasing monotonically, \( r_{AD}(t) \geq 0 \) for \( r_X(t) \geq 0 \).

In slight contrast to the DDP, the lifetime failure rate corresponding to the IDP is
\[
r_{AI}(t) = -\log\left[G_X(\Df - \eta(t))\right] = \eta'(t) \cdot \frac{g_X(\Df - \eta(t))}{G_X(\Df - \eta(t))},
\]
with
\[
R_{AI}(t) = -\log\left[G_X(\Df - \eta(t))\right].
\]
The \( p^{th} \) quantile of lifetime distribution with the DDP is the unique value \( t_p \) that satisfies \( F_{AD}(t_p) = G_X(\Df - \eta(t_p)) = p \), and assuming that \( \eta \) is monotonic and continuous with inverse \( \eta^{-1} \),
\[
t_{AD,p} = \eta^{-1}[\Df - G_X^{-1}(p)],
\]
and the corresponding \( p^{th} \) quantile of the IDP is
\[
t_{AI,p} = \eta^{-1}[\Df - G_X^{-1}(1 - p)].
\]

### 2.2 Multiplicative Degradation Model

In some cases, multiplicative models are exchangeable with additive models through a log transformation; for example, when \( X \) is a lognormal-distributed random variable in the multiplicative degradation model, log-\( X \) has a normal distribution in the additive degradation model. In cases where \( \eta \) contains random effects, however, the multiplicative model is not interchangeable an additive model. The general multiplicative degradation model can be expressed as
\[
\D(t; X, \Theta) = X \cdot \eta(t; \Theta).
\]

Failure-time distributions corresponding to the DDP and the IDP are
\[
F_{MD}(t) = Pr \left[ X < \frac{\Df}{\eta(t)} \right] = G_X \left( \frac{\Df}{\eta(t)} \right)
\]
\[
F_{MI}(t) = 1 - G_X \left( \frac{\Df}{\eta(t)} \right). \]

To satisfy \( F_{MI}(0-) = 0 \) for IDP in the multiplicative degradation model, we need \( G_X(\frac{\Df}{\eta(t)})|_{t \to 0-} \to 1 \). The failure rate for \( F_{MI} \) in (9) is
\[
r_{MD}(t) = -\left( \frac{\Df}{\eta(t)} \right) \cdot \log[\eta(t)] \cdot r_X \left( \frac{\Df}{\eta(t)} \right)
\]
and for the lifetime distribution based on the IDP, the failure rate is
\[
r_{MI}(t) = \left( \frac{\Df}{\eta(t)} \right) \cdot \log[\eta(t)] \cdot \frac{g_X(\frac{\Df}{\eta(t)})}{G_X(\frac{\Df}{\eta(t)})}.
\]

Similar to the additive model, the failure rates for the multiplicative model are scaled by the failure rate for the degradation model. The \( p^{th} \) quantile for \( F_{MD}(t) \) and \( F_{MD}(t) \) are
\[
t_{MD,p} = \eta^{-1} \left[ \frac{\Df}{G_X^{-1}(p)} \right],
\]
\[
t_{MI,p} = \eta^{-1} \left[ \frac{\Df}{G_X^{-1}(1 - p)} \right].
\]

### 3. Reliability Characteristics of Implied Lifetime Distributions

In this section, we investigate properties of lifetime distributions generated from the additive and multiplicative degradation models. The lifetime distribution is uniquely determined by the degradation distribution’s failure rate, and accordingly, we provide
basic definitions for the classes of a lifetime distribution in terms of failure rate.

**Definition 3.1** Lifetime distribution $F$ is an increasing failure rate (IFR) distribution if its failure rate $r(t)$ increases monotonically over time; that is, $r'(t) \geq 0$ for all $t \geq 0$. Likewise, $F$ has a decreasing failure rate (DFR) if $r'(t) \leq 0$.

**Definition 3.2** Lifetime distribution $F$ is defined as increasing (decreasing) failure rate average (IFRA) or (DFRA) if $-\log(F(t))/t = \int_0^t r(u)du/t$ is nondecreasing (nonincreasing) in $t$ on $\{t \in \mathbb{R}_+ : F(t) > 0\}$.

**Definition 3.3** Lifetime distribution $F$ is bathtub (BT) or upside-down bathtub (UBT) shaped, if there exists $0 \leq t_1 \leq t_2 \leq \infty$ such that $r(t)$ strictly decreases (that is, $r'(t) < (>) 0$) if $0 \leq t < t_1$, is approximately constant ($r'(t) \approx 0$) if $t_1 \leq t \leq t_2$, and strictly increases ($r'(t) > (>) 0$) if $t_2 < t \leq \infty$.

Obviously, the IFR class is a subset of the IFRA class. A bathtub failure rate characterizes life tests in which some early failures occur for a short time until failure rate stabilizes, then eventually increases as the test item ages.

### 3.1 Additive Degradation Model

Firstly define $\alpha(t)$ as the reciprocal of the failure rate,

$$\alpha(t) = r(t)^{-1} = \frac{\bar{F}(t)}{f(t)}.$$ 

It follows that $\alpha(t)$ is positive, continuous, and twice differentiable on $(0, \infty)$. For the DDP in an additive degradation model, we have

$$\alpha_{AD}'(t) = -\left[\eta'(t) \cdot r_X(D_{\theta} - \eta(t))^{-1}\right]' = \alpha_{AD}(t) \left[\eta'(t) \xi_X(t) - \eta''(t)\right],$$

where $\xi_X(t)$ is defined by

$$\xi_X(t) = \frac{r_X'(D_{\theta} - \eta(t))}{r_X(D_{\theta} - \eta(t))}.$$ 

It can be easily proven that the lifetime distribution derived from the DDP has DFR since $\xi_X(t) \leq 0$ for the increasing function $D_{\theta} - \eta(t)$. Consequently, $\alpha_{AD}'(t) > 0$ for all $t \geq 0$, which, from (15), implies that $F_{AD}(t)$ is a DFR distribution. More conditions are necessary in order for $F_{AD}(t)$ to possess DFR.

**Theorem 3.1** For the additive degradation model with decreasing $\eta(t)$ with random error $X \sim G$, if $\xi_X(t)$ is bounded with lower limit $-(d/dt)\eta(t)^{-1}$, then $F_{AD}(t)$ has increasing failure rate.

**Proof:** $F_{AD}(t)$ has IFR $\iff \alpha_{AD}'(t) < 0 \iff \xi_X(t) \geq \eta''(t)/\eta'(t)^{-2} = -(d/dt)\eta'(t)^{-1}.$

**Theorem 3.2** If $G_X(t)$ is an IFR distribution, then $F_{AD}(t)$ possesses IFR.

**Proof:** The IFR property implies that $R_X(T) = -\log G_X(T)$ is convex, that is, $R_X(\gamma T) \leq \gamma R_X(T)$. Since $D_{\theta} - \eta(t)$ is an increasing function, by taking $T = D_{\theta} - \eta(t)$, $R_X(\gamma(D_{\theta} - \eta(t))) \leq \gamma R_X(D_{\theta} - \eta(t)) \iff R_{AD}(\gamma t) \leq \gamma R_{AD}(t) \iff F_{AD}(\gamma t) \geq [F_{AD}(t)]^{\gamma}$. This is equivalent to the IFRA property.

A bathtub-shaped failure rate for the lifetime distribution can be generated from the additive model. Following Theorem 3.1, it can be shown that $F_{AD}(t)$ holds BT (or UBT) shaped failure rate provided that there exists $t^* > 0$ such that $\xi_X(t) < (>) -(d/dt)\eta'(t)^{-1}$ for all $t \in (0, t^*)$. 

$\xi_X(t^*) = -(d/dt)\eta'(t)^{-1}|_{t=t^*} = 0$, and $\xi_X(t) > (>) -(d/dt)\eta'(t)^{-1}$ for all $t > t^*$.

### 3.2 Multiplicative Degradation Model

The failure rates of lifetime distributions derived from the multiplicative degradation model, as given in (11) for the DDP and (12) for the IDP, depend directly on the deterministic degradation function $\eta(t)$. For example, consider the following DDP

$$D(t; X, \Theta) = X \cdot (\theta_1 t + 1)^{-\theta_2}, \quad \theta_1, \theta_2 > 0, \quad (17)$$

where $X$ follows a Weibull distribution with a scale parameter $\lambda > 0$, and a shape parameter $\kappa > 0$. As shown in the Figure 1, failure rates at fixed values of $\theta_1 = 0.1$ and $\lambda = 1$ are decreasing ($\theta_2 < 2.0$), constant ($\theta_2 = 2.0$), or increasing ($\theta_2 > 2.0$) even if $X$ has DFR ($\kappa = 0.5$).

In this example, the failure rate of the IDP eventually decreases to zero, and this phenomenon can be easily explained in that $\eta(t) \gg [\log[\eta(t)]]'$, consequently

$$r_{mt}(t) \approx g_X(\frac{D_{\theta}}{\eta(t)}) \cdot \frac{g_X(\frac{D_{\theta}}{\eta(t)})}{G_X(\frac{D_{\theta}}{\eta(t)})} \to 0$$

as $t \to \infty$. From simulation results, it could be observed that resulting failure rate from the IDP
Failure rate

0.025 0.030 0.035 0.040

where

X

Gaussian process is defined as the stochastic process

tic process, specifically the

result of performance degradation using a stochas-

tic process in modeling random fatigue growth.

be modelled as a stochastic process. Sobczyk and

function is the Paris-Erdogan relationship \([5]\),

cumulation. A commonly used crack growth rate

chanics,

\(\eta\)

is the stress intensity range, which is a function of

\(\alpha\)

where

\(\alpha\) is a measure of the correlation between

\(X(t)\) and \(X(t + \tau)\). This autocovariance provides

flexibility in covering a wide range of dispersion of

crack growth accumulation through the correlation

parameter \(\zeta\). Based on the mean crack growth rate

\(\eta(\Delta K(\alpha))\) and the lognormal random process \(X(t)\)

with the autocovariance in (19), we can derive a life-
time distribution of a fatigue degradation process.

Denote a deterministic initial crack size as \(a_0\) at

t_0 = 0 and the crack size at service time \(\tau\) as \(a(\tau)\).

Then the mean service time to reach \(a(\tau)\) from \(a_0\) is

\[
\int_{a_0}^{a(\tau)} \frac{d\alpha}{\eta(\Delta K(\alpha))} = \int_0^\tau X(t)dt,
\]

because \(K(\alpha)\) is a function of the crack size \(\alpha\). Let

\(\omega(\tau) = \int_0^\tau X(t)dt\), then the crack size \(a(\tau)\) is a

monotone increasing function of \(\omega(\tau)\). The integra-
tion of the lognormal random process, \(\{\omega(\tau) : \tau > 0\}\)

is also a lognormal random process with mean

\[
E[\omega(\tau)] = \int_0^\tau E[X(t)]dt = \mu_X \tau,
\]

and variance

\[
Var[\omega(\tau)] = \int_0^\tau \int_0^\tau \sigma^2_X \cdot \exp(-\zeta(t_2 - t_1))dt_1dt_2
\]

\[= 2 \left( \frac{\sigma^2_X}{\zeta} \right)^2 \cdot (\exp(-\zeta \tau) + \zeta - 1).\]

For a fatigue process, lifetime is defined as the time

that a crack size increases beyond a pre-determined

threshold value \(\alpha_f\), therefore

\[
Pr[a(\tau) \geq \alpha_f] = Pr[\omega(\tau) \geq \omega_f]
\]

\[= 1 - \Phi \left( \frac{\log \omega_f - \mu_\omega(\tau)}{\sigma_\omega(\tau)} \right)\]

(20)
where $\omega_f = \int_{\alpha_0}^{\alpha} (\eta(\Delta K(\alpha)))^{-1} d\alpha$, and $\mu_\omega(\tau), \sigma_\omega(\tau)$ can be obtained by solving the following equations

$$E[\omega(\tau)] = \exp[\mu_\omega(\tau) + \sigma_\omega^2(\tau)/2]$$

and

$$\text{Var}[\omega(\tau)] = \exp[2\mu_\omega(\tau) + \sigma_\omega^2(\tau)] \cdot (\exp[\sigma_\omega^2(\tau)] - 1).$$

The lifetime distribution presented in (20) is flexible enough to cover a broad domain of fatigue growth failures by taking into account the correlation parameter $\zeta$. In the case the correlation parameter $\zeta$ approaches zero, for example, from (19) the autocovariance function $\gamma_X(\tau)$ is independent of $\tau$ and (20) is reduced to a lognormal distribution for $\omega(\tau) = X\tau$. For the Paris-Erdogan relationship, the failure rate of (20) is decreasing in $\tau$.

5. Conclusion

In modeling material fatigue and degradation, the relationship between the randomness in degradation and randomness in the resulting lifetime distribution is strong and direct, albeit hard to discern. By making tacit assumptions about the degradation distribution, the resulting implications to the lifetime distribution may surprise the experimenter and possibly contradict the assumptions about the failure characteristics in the study.

The comparisons in this article are based on basic degradation functions that are applied to show the relationships between degradation and lifetime with minimal complexity. Certainly, most practical applications would use degradation functions extended beyond the elementary ones listed here. However, these results bolster the need for further attention to this implied relationship of interest; few efforts have been pursued toward this end.

While degradation modeling is growing beyond the purview of manufacturing and materials testing, reliability prediction through degradation modeling will be further emphasized as a supporting tool in lifetime data analysis.

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References


