Degradation models and implied lifetime distributions

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Abstract

In experiments where failure times are sparse, degradation analysis is useful for the analysis of failure time distributions in reliability studies. This research investigates the link between a practitioner’s selected degradation model and the resulting lifetime model. Simple additive and multiplicative models with single random effects are featured. Results show that seemingly innocuous assumptions of the degradation path create surprising restrictions on the lifetime distribution. These constraints are described in terms of failure rate and distribution classes.

Keywords: Additive model; Bathtub and increasing failure rates; Random effects; Stochastic ordering

1. Introduction

Reliability tests based on time-to-failure measurements are often hampered by the lack of observed failures. Accelerated life testing (ALT) can hasten product failure during test intervals by stressing the product beyond its normal use. In many tests, the failure data are supplemented by degradation data, which might be measurements of product wear available at one or more time points during the reliability test. The product lifetime is defined to be the first point in time when the degradation reaches a pre-specified threshold value.

The collection of degradation data has become necessary in many industries because the highly reliable units on test show few if any failures over the limited test period. Meeker and Escobar [1] offer a comprehensive guide to degradation analysis for various life tests, including ALT, and show that degradation analysis has great potential to improve upon reliability analysis.

However, degradation analysis can also introduce the potential for inconsistency in the experimenter’s treatment of the data. The key to the analysis is the perceived link between the degradation measurements and the failure time.

By assuming a stochastic model for the degradation, the lifetime distribution is automatically implied, and in many cases, these implied lifetime distributions are unwieldy and do not match the assumptions of the experimenter. Typically, the resulting estimate of the lifetime distribution must be solved numerically with estimate uncertainty computed using simulation and intensive re-sampling methods such as bootstrap procedures.

In this article, we investigate the link between a chosen degradation model and the resulting lifetime model. We consider simple additive and multiplicative models and seek degradation models that lead to particular families (e.g., increasing failure rate (IFR) or “bathtub” (BT) shaped failure rate) of lifetime distributions.

2. Degradation models

Degradation models vary markedly across the fields of reliability modeling. Many practical problems can be modeled with a linear (or log-linear) rate of degradation, such as Lu et al. [2] random effects model for semiconductor degradation. Bogdanoff and Kozin [3] employ both linear and more complex nonlinear models to characterize...
degradation in materials testing (e.g., crack growth). The random effects in the degradation model are the key link between the degradation function and the resulting lifetime distribution. For example, the Bernstein distribution describes the failure-time distribution for a simple linear model with random intercept and random slope [4]. Some degradation models employ a single random effect as an error term in an additive model. Contemporary degradation models are apt to consider several random effects that enter into the degradation function in nonlinear form, including multiplicative terms [5]. For this research, only basic additive and multiplicative models are considered, and the focus is on a single random effect. For simplicity, we assume the item-to-item variation (characterized by this random effect) dominates the variability in degradation, and we ignore error terms, such as within-item variability.

2.1. Additive degradation model

Consider the general additive degradation model

$$\tilde{D}(t; X, \Theta) = \eta(t; \Theta) + X,$$

where $\eta(t; \Theta)$ is a deterministic mean degradation path with fixed effect parameters $\Theta$ for time $t \geq 0$. We focus on $\eta$ being monotonic since most degradation measurements have this quality. Bae and Kvam [6] consider the nonmonotonic degradation of light displays, but this example is the exception to the norm. $X$ represents random variation around a mean degradation level $\eta(t; \Theta)$ with a cumulative distribution function (cdf) $G_X$ and a probability density function (pdf) $g_X$.

We assume that failure occurs when the test item’s degradation level reaches a predetermined threshold value ($\tilde{D}_f$). For a monotonically decreasing degradation path (DDP), a failure is defined as the time that the degradation level decreases below the threshold, i.e., $\tilde{D}(t; X, \Theta) < \tilde{D}_f$ and $\tilde{D}(t; X, \Theta) > \tilde{D}_f$ for a monotonically increasing degradation path (IDP). Obviously, the distribution of lifetime is achieved through a transformation of the random effects distribution, and is largely a function of the mean degradation function $\eta$.

Let $F_{AD}(t)$ denote the lifetime distribution function generated by the DDP in the additive degradation model (1). Then

$$F_{AD}(t) = \Pr[\tilde{D}(t; X, \Theta) < \tilde{D}_f] = \Pr[X < \tilde{D}_f - \eta(t; \Theta)]$$

$$= G_X(\tilde{D}_f - \eta(t; \Theta)), \quad \text{(2)}$$

with survival function $\bar{F}_{AD}(t) = 1 - F_{AD}(t) = 1 - G_X(\tilde{D}_f - \eta(t; \Theta))$. For the IDP, the lifetime distribution is $F_{AI}(t) = 1 - G_X(\tilde{D}_f - \eta(t; \Theta))$. If $T$ is the lifetime, then this model defines the transformation to $T = T(X)$.

Note that (2) is a valid distribution only if $G_X(\tilde{D}_f - \eta(0; \Theta)) = 0$ and $G_X(\tilde{D}_f - \eta(+\infty; \Theta)) = 1$. For the IDP, we require that $G_X(\tilde{D}_f - \eta(0; \Theta)) = 1$ and $G_X(\tilde{D}_f - \eta(+\infty; \Theta)) = 0$. If $\eta(t; \Theta)$ has finite asymptotes, for example, the additive degradation model will not necessarily produce a proper lifetime distribution function. Along with these constraints on $\eta$, we assume $G$ (and hence $F$) is twice differentiable on $(0, \infty)$. Hereafter, we write $\eta(t)$ instead of $\eta(t; \Theta)$ for simplicity. Though this simple relationship between $T$ and $X$ is plain to the reader, its effect on the lifetime properties is more hidden. This can be shown through the failure rate function.

The failure rate corresponding to the DDP, $r_{AD}(t)$, is defined by

$$r_{AD}(t) = \frac{f_{AD}(t)}{1 - F_{AD}(t)} = \frac{[G_X(\tilde{D}_f - \eta(t))]'}{1 - G_X(\tilde{D}_f - \eta(t))}$$

$$= -\eta'(t) \cdot \frac{g_X(\tilde{D}_f - \eta(t))}{1 - G_X(\tilde{D}_f - \eta(t))}$$

$$= -\eta'(t) \cdot r_X(\tilde{D}_f - \eta(t)), \quad \text{(3)}$$

where $r_X$ denotes the failure rate of the degradation random variable $X$. If $\eta(t)$ is continuous and decreasing monotonically, $r_{AD}(t) \geq 0$ for $r_X(t) \geq 0$. Note that failure rate of the DDP is closely related to that of the random effect $X$ and is greatly affected by the functional form of $\eta(t)$. The cumulative failure rate is $R_{AD}(t) = \int_0^t r_{AD}(x) \, dx$, and can be expressed as $R_{AD}(t) = -\log \bar{F}_{AD}(t) = -\log[1 - G_X(\tilde{D}_f - \eta(t))]$.

In slight contrast to the DDP, the lifetime failure rate corresponding to the IDP is

$$r_{AI}(t) = -\frac{[G_X(\tilde{D}_f - \eta(t))]'}{G_X(\tilde{D}_f - \eta(t))} = \eta'(t) \cdot \frac{g_X(\tilde{D}_f - \eta(t))}{G_X(\tilde{D}_f - \eta(t))}$$

$$\quad \text{(4)}$$

with

$$R_{AI}(t) = -\log[G_X(\tilde{D}_f - \eta(t))]. \quad \text{(5)}$$

The $p$th quantile of lifetime distribution with the DDP is the unique value $t_p$ that satisfies $F_{AD}(t_p) = G_X(\tilde{D}_f - \eta(t_p)) = p$, and assuming that $\eta$ is monotonic and continuous with inverse $\eta^{-1}$,

$$t_{AD,p} = \eta^{-1}[G_X(\tilde{D}_f - G_X^{-1}(p))], \quad \text{(6)}$$

and the corresponding $p$th quantile of the IDP is

$$t_{AI,p} = \eta^{-1}[\tilde{D}_f - G_X^{-1}(1 - p)]. \quad \text{(7)}$$

2.2. Multiplicative degradation model

In some cases, multiplicative models are exchangeable with additive models through a log transformation; for example, when $X$ is a lognormal-distributed random variable in the multiplicative degradation model, $\log X$ has a normal distribution in the additive degradation model. However, in this section we consider general cases where the multiplicative model is not interchangeable with an additive model. The general multiplicative degradation model can be expressed as

$$\tilde{D}(t; X, \Theta) = X \cdot \eta(t; \Theta). \quad \text{(8)}$$
Failure-time distributions corresponding to the DDP and the IDP are
\[ F_{\text{MD}}(t) = \Pr[X \cdot \eta(t) < \mathcal{D}_t] = \Pr \left[ X < \frac{\mathcal{D}_t}{\eta(t)} \right] = G_X \left( \frac{\mathcal{D}_t}{\eta(t)} \right), \]
(9)
\[ F_{\text{MI}}(t) = 1 - G_X \left( \frac{\mathcal{D}_t}{\eta(t)} \right). \]
(10)
To satisfy \( F_{\text{MI}}(0^-) = 0 \) for an IDP in the multiplicative degradation model, we need \( G_X(\mathcal{D}_t/\eta(0)) = 1 \), and similarly \( \lim_{t \to +\infty} G_X(\mathcal{D}_t/\eta(t)) = 0 \) to satisfy \( F_{\text{MI}}(+\infty) = 1 \). The failure rate for \( F_{\text{MD}} \) in (9) is
\[ r_{\text{MD}}(t) = \frac{\left[ G_X(\mathcal{D}_t/\eta(t)) \right]'}{1 - G_X(\mathcal{D}_t/\eta(t))} = - \left[ \frac{\mathcal{D}_t}{\eta(t)} \right]' \log[\eta(t)]' \cdot r_X \left( \frac{\mathcal{D}_t}{\eta(t)} \right), \]
(11)
and for the lifetime distribution based on the IDP, the failure rate is
\[ r_{\text{MI}}(t) = - \frac{\left[ G_X(\mathcal{D}_t/\eta(t)) \right]'}{G_X(\mathcal{D}_t/\eta(t))} = \left( \frac{\mathcal{D}_t}{\eta(t)} \right)' \log[\eta(t)]' \cdot r_X(\mathcal{D}_t/\eta(t)). \]
(12)
Similar to the additive model, the failure rate for the multiplicative model is scaled by the failure rate for the degradation model. The \( p \)-th quantile for \( F_{\text{MD}}(t) \) and \( F_{\text{MI}}(t) \) are, respectively,
\[ t_{\text{MD},p} = \eta^{-1} \left[ \mathcal{D}_t \left( \frac{G_X(p)}{G_X(1-p)} \right) \right], \]
(13)
\[ t_{\text{MI},p} = \eta^{-1} \left[ \mathcal{D}_t \left( \frac{1}{G_X(1-p)} \right) \right]. \]
(14)

3. Reliability characteristics of implied lifetime distributions

In this section, we investigate properties of lifetime distributions generated from the additive and multiplicative degradation models. The lifetime distribution is uniquely determined by the degradation distribution’s failure rate, and accordingly, we provide basic definitions for the classes of a lifetime distribution in terms of failure rate.

**Definition 3.1.** Lifetime distribution \( F \) is an IFR distribution if its failure rate \( r(t) \) increases monotonically over time; that is, \( r'(t) \geq 0 \) for all \( t \geq 0 \). Likewise, \( F \) has a decreasing failure rate (DFR) if \( r'(t) \leq 0 \).

**Definition 3.2.** Lifetime distribution \( F \) is defined as increasing (decreasing) failure rate average (IFRA) or (DFRA) if \( -\log(F(t))/t = \int_0^t r(u) \, du/t \) is nondecreasing (nonincreasing) in \( t \) on \( \{ t \in \mathbb{R}_+ : F(t) > 0 \} \).

**Definition 3.3.** The failure rate of the lifetime distribution \( F \) is BT shaped if there exists \( 0 \leq t_1 \leq t_2 \leq \infty \) such that \( r(t) \) strictly decreases (that is, \( r'(t) < 0 \)) if \( t < t_1 \), is approximately constant if \( t_1 \leq t \leq t_2 \), and strictly increases \( (r'(t) > 0) \) if \( t > t_2 \).

**Definition 3.4.** The failure rate of the lifetime distribution \( F \) is upside-down bathtub (UBT) shaped if there exists \( 0 \leq t_1 \leq t_2 \leq \infty \) such that \( r(t) \) strictly increases (that is, \( r'(t) > 0 \)) if \( t < t_1 \), is approximately constant if \( t_1 \leq t \leq t_2 \), and strictly decreases \( (r'(t) < 0) \) if \( t > t_2 \).

Obviously, the IFR class is a subset of the IFRA class. A BT failure rate characterizes life tests in which some early failures occur for a short time until failure rate stabilizes, then eventually increases as the test item ages.

To illustrate the relationship between the lifetime distribution and the degradation distribution, we first need to define \( x(t) \) as the reciprocal of the failure rate:
\[ x(t) = r(t)^{-1} = \frac{\bar{F}(t)}{f(t)}. \]
(15)
It follows that \( x(t) \) is positive, continuous, and twice differentiable on \((0, \infty)\). For the DDP in an additive degradation model, we have
\[ x_{\text{AD}}'(t) = - \left[ (\eta'(t) \cdot r_X(\mathcal{D}_t - \eta(t)))^{-1} \right]' \]
\[ = x_{\text{AD}}(t) \left[ \eta'(t) \xi_X(t) - \eta'(t) \right], \]
(16)
where \( \xi_X(t) \) is defined by
\[ \xi_X(t) = \frac{r_X'(\mathcal{D}_t - \eta(t))}{r_X(\mathcal{D}_t - \eta(t))}. \]
(17)
Both \( x \) and \( \xi \) are useful to show the following theorems.

**Theorem 3.1.** For the additive degradation model with decreasing \( \eta(t) \) and random error \( X \sim G \), the lifetime distribution derived from the DDP has DFR if \( G_X(t) \) possesses DFR and \( \eta(t) \) is strictly convex.

**Proof.** Since \( G_X(t) \) possesses DFR, \( \xi_X(t) \leq 0 \) for the increasing function \( \mathcal{D}_t - \eta(t) \), and \( \eta'(t)/\eta(t) \leq 0 \) for convex decreasing function \( \eta(t) \). Consequently, \( x_{\text{AD}}'(t) > 0 \) for all \( t \geq 0 \), which, from (15), implies that \( F_{\text{AD}}(t) \) is a DFR distribution.

**Theorem 3.2.** The lifetime distribution \( F_{\text{AD}}(t) \) has IFR if \( G_X(t) \) possesses IFR and \( \eta(t) \) is strictly concave.

**Proof.** \( x_{\text{AD}}(t) \geq 0 \) for \( G_X(t) \) possessing IFR with respect to the increasing function \( \mathcal{D}_t - \eta(t) \), and \( \eta'(t)/\eta(t) \geq 0 \) for concave decreasing function \( \eta(t) \). Consequently, \( x_{\text{AD}}'(t) < 0 \) for all \( t > 0 \), which implies that \( F_{\text{AD}}(t) \) is a IFR distribution.

**Theorem 3.3.** If \( G_X(t) \) is an IFR distribution, then \( F_{\text{AD}}(t) \) possesses IFRA.

**Proof.** The IFR property implies that \( R_X(T) = -\log \tilde{G}_X(T) \) is convex, that is, \( R_X(y) \leq \gamma R_X(T) \). Since
\( \mathcal{D}_t - \eta(t) \) is an increasing function, by taking \( T = \mathcal{D}_t - \eta(t) \), \( R_X(y(\mathcal{D}_t - \eta(t))) \leq \gamma R_X(\mathcal{D}_t - \eta(t)) \iff R_{\text{IDP}}(t) \leq \gamma R_{\text{DDP}}(t) \iff F_{\text{IDP}}(t) \geq F_{\text{DDP}}(t). \) This is equivalent to the IFRA property.

Similar results hold if the degradation path is increasing, rather than decreasing. Consider the IDP in an additive degradation model. By plugging (4) into (15) and differentiating with respect to \( t \),

\[
x'_{\text{AI}}(t) = \frac{g_X(\mathcal{D}_t - \eta(t))}{g_X(\mathcal{D}_t - \eta(t))} \left[ -\frac{\eta''(t)}{\eta'(t)^2} + \delta(t) - 1 \right],
\]

where

\[
\delta(t) = \frac{g_X(\mathcal{D}_t - \eta(t))}{g_X(\mathcal{D}_t - \eta(t))}.
\]

Noting that \( G_X(\mathcal{D}_t - \eta(t)) = -\int_0^t \eta'(y) \cdot g_X(\mathcal{D}_t - \eta(y)) \, dy \), where \( t_0 = \inf \{ t : G_X(\mathcal{D}_t - \eta(t)) > 0 \} \), \( x'_{\text{AI}}(t) \) can be represented as

\[
x'_{\text{AI}}(t) = \int_{t_0}^t \frac{-\eta'(y)g_X(\mathcal{D}_t - \eta(y))}{g_X(\mathcal{D}_t - \eta(t))} \left[ -\frac{\eta''(t)}{\eta'(t)^2} + \delta(t) - \delta(y) \right] \, dy
\]

\[
+ \int_{t_0}^t \frac{-\eta'(y)g_X(\mathcal{D}_t - \eta(y))}{g_X(\mathcal{D}_t - \eta(t))} \cdot \delta(y) \, dy - 1.
\]

However,

\[
\int_{t_0}^t \frac{-\eta'(y)g_X(\mathcal{D}_t - \eta(y))}{g_X(\mathcal{D}_t - \eta(t))} \cdot \delta(y) \, dy
\]

\[
= \int_{t_0}^t \frac{-\eta'(y)g_X(\mathcal{D}_t - \eta(y))}{g_X(\mathcal{D}_t - \eta(t))} \, dy = \frac{g_X(\mathcal{D}_t - \eta(t))}{g_X(\mathcal{D}_t - \eta(t))} = 1,
\]

and consequently,

\[
x'_{\text{AI}}(t) = \int_{t_0}^t \frac{-\eta'(y)g_X(\mathcal{D}_t - \eta(y))}{g_X(\mathcal{D}_t - \eta(t))} \left[ \delta(y) - \delta(t) + \frac{\eta''(t)}{\eta'(t)^2} \right] \, dy,
\]

(19)

which leads to the following result.

**Theorem 3.4.** For a strictly convex increasing degradation path \( \eta(t) \), if \( \delta(t) = \frac{g_X(\mathcal{D}_t - \eta(t))}{g_X(\mathcal{D}_t - \eta(t))} \) decreases monotonically, then \( F_{\text{AI}}(t) \) has DFR.

**Proof.** Based on (19), if \( \delta'(t) < 0 \), then \( x'_{\text{AI}}(t) > 0 \) for all \( t \geq 0 \), which implies that \( F_{\text{AI}}(t) \) is a DFR distribution.

The result in (19) will be useful to define sufficient conditions for a BT or UBT shaped failure rate of \( F_{\text{AI}}(t) \).

The failure rates of lifetime distributions derived from the multiplicative degradation model, as given in (11) for the DDP and (12) for the IDP, depend directly on the deterministic degradation function \( \eta(t) \). For example, consider the DDP

\[
\mathcal{D}(t; X, \Theta) = X \cdot (\theta_1 t + 1)^{-\theta_2}, \quad \theta_1, \theta_2 > 0,
\]

(20)

where \( X \) follows a Weibull distribution with a scale parameter \( \lambda > 0 \), and a shape parameter \( \kappa > 0 \). As shown in Fig. 1, failure rates at fixed values of \( \theta_1 = 0.1 \) and \( \lambda = 1.0 \) are decreasing \( (\theta_2 < 2.0) \), constant \( (\theta_2 = 2.0) \), or increasing \( (\theta_2 > 2.0) \) even if \( X \) has DFR \( (\kappa = 0.5) \).

The failure rate of the IDP eventually decreases to zero, and this phenomenon can be easily explained by the fact that \( \eta(t) \approx [\log(\eta(t))]' \). Consequently

\[
r_{\text{MT}}(t) \approx \frac{[\log(\eta(t))]'}{\eta(t)} \frac{g_X(\mathcal{D}_t/\eta(t))}{G_X(\mathcal{D}_t/\eta(t))} \to 0
\]

as \( t \to \infty \).

It can be observed that the resulting failure rate from the IDP is a decreasing or unimodal function, depending to parameters of the random effect distribution and the form of \( \eta(t) \). For example, the degradation path (20) with \( \theta_2 < 0 \) is monotonically increasing. Using this with a Weibull distributed \( X \) having fixed scale value \( (\lambda = 1) \), the failure rate is unimodal \( (\kappa = 2.0) \) or decreasing \( (\kappa = 0.5) \) when \( \theta_1 = 0.1 \) and \( \theta_2 = 2.0 \) (see Fig. 2).

4. **Stochastic ordering of a degradation lifetime distribution**

Denote the distribution function and survival function of a random variable \( X_i \) by \( F_i \) and \( S_i \), respectively, for \( i = 1, 2 \). In this section, we employ stochastic orders
between degradation distributions to derive properties for
the implied lifetime distribution.

**Definition 4.1.** \( X_1 \) is said to be **stochastically smaller** than \( X_2 \), (written \( X_1 \preceq_{s} X_2 \)) if \( F_1(t) \preceq F_2(t) \) for all \( t \). Equivalently, \( X_1 \preceq_{s} X_2 \) if \( E[\psi(X_1)] \leq E[\psi(X_2)] \) for any increasing, integrable function \( \psi(\cdot) \).

**Definition 4.2.** Assume \( X_1 \) and \( X_2 \) are absolutely continuous random variables with density \( f_1(t) \) and \( f_2(t) \). \( X_1 \) is smaller than \( X_2 \) in terms of likelihood ratio (written \( X_1 \preceq_{lr} X_2 \)) if \( f_1(t)/f_2(t) \) is nonincreasing in \( t \).

It is well-known that \( X_1 \preceq_{lr} X_2 \) implies that \( X_1 \preceq_{s} X_2 \) [7].

**Theorem 4.1.** Let \( X_1 \) and \( X_2 \) be the random effects of corresponding degradation paths \( \varphi_1 \) and \( \varphi_2 \). For the DDPS (IDPs) \( \varphi_1 \) and \( \varphi_2 \) having \( \eta(t) > 0 \) of the multiplicative form, we have

\[ \begin{align*}
& (i) \text{ If } X_1 \preceq_{s} X_2, \text{ then } \varphi_1 \preceq_{s} (\geq_{s}) \varphi_2. \\
& (ii) \text{ If } X_1 \preceq_{lr} X_2, \text{ then } \varphi_1 \preceq_{lr} (\geq_{lr}) \varphi_2. \\
& (iii) \text{ If } \varphi_1 \preceq_{lr} \varphi_2, \text{ then } X_1 \preceq_{s} (\geq_{s}) X_2.
\end{align*} \]

**Proof.** For (i),
\[
\Pr[X_1 \leq x] \geq \Pr[X_2 \leq x] \\
\iff \Pr[X_1, \eta(t) \leq \varphi_1] \geq \Pr[X_2, \eta(t) \leq \varphi_1], \text{ for } \eta(t) > 0 \\
\iff \Pr[\varphi_1 \leq \varphi_1] \geq \Pr[\varphi_2 \leq \varphi_1] \\
= \frac{F_{MD_1}}{F_{MD_2}}(F_{ML_1}(\geq_{s})F_{ML_2}).
\]
which implies that \( \varphi_1 \preceq_{s} (\geq_{s}) \varphi_2 \),

Using the DDPS \( \varphi_1 \) and \( \varphi_2 \) in (ii), a partial ordering of \( X_1 \) and \( X_2 \) can be translated into a partial ordering on an increasing function of \( t, \varphi_1(\eta(t)) \geq 0 \), where \( \varphi_1(t) \geq 0 \) is a constant and \( t \geq 0 \). Therefore,
\[
\begin{pmatrix} G_{X_1}(x) & g_{X_1}(x) \\ G_{X_2}(x) & g_{X_2}(x) \end{pmatrix} \downarrow x \iff \begin{pmatrix} G_{X_1}(\varphi_1(\eta(t))) & g_{X_1}(\varphi_1(\eta(t))) \\ G_{X_2}(\varphi_1(\eta(t))) & g_{X_2}(\varphi_1(\eta(t))) \end{pmatrix} \downarrow t \\
\iff \begin{pmatrix} F_{MD_1}(t) & f_{MD_1}(t) \\ F_{MD_2}(t) & f_{MD_2}(t) \end{pmatrix} \downarrow t.
\]

Similarly for the IDPs \( \varphi_1 \) and \( \varphi_2 \), a partial ordering of \( X_1 \) and \( X_2 \) can be reversed translated into a partial ordering on a decreasing function of \( t, \varphi_1(\eta(t)) \geq 0 \), therefore
\[
\begin{pmatrix} G_{X_1}(x) & g_{X_1}(x) \\ G_{X_2}(x) & g_{X_2}(x) \end{pmatrix} \downarrow x \iff \begin{pmatrix} G_{X_1}(\varphi_1(\eta(t))) & g_{X_1}(\varphi_1(\eta(t))) \\ G_{X_2}(\varphi_1(\eta(t))) & g_{X_2}(\varphi_1(\eta(t))) \end{pmatrix} \uparrow t \\
\iff \begin{pmatrix} F_{ML_1}(t) & f_{ML_1}(t) \\ F_{ML_2}(t) & f_{ML_2}(t) \end{pmatrix} \uparrow t.
\]

See Keilson and Sumita [8] for detailed proof of (iii).

5. **Reliability ordering of a degradation lifetime distribution**

Partial orderings of life distributions in terms of their aging properties were introduced by Kochar and Wiens [9]. We will employ some of those definitions, summarized below, to investigate how the stochastic orderings affect comparisons in the lifetime distributions. The orderings are based on the previously defined stochastic orders IFR, IFRA and another ordering called “New Better than Used” (NBU). A distribution \( F \) is NBU iff
\[
\tilde{F}(x) \tilde{F}(y) \geq F(x + y)
\]
for all values of \( x \) and \( y \). More intuitively, this is equivalent to \( P(X > t) \geq P(X > t + y | X > y) \). Note that \( F \in \text{IFRA} \Rightarrow F \in \text{NBU} \).

**Definition 5.1.** \( F_1 \) is more IFR than \( F_2 \) (written \( F_1 <_{\text{IFR}} F_2 \)) if \( F_2^{-1}(F_1(t)) \) is a convex function in \( t \) on the support of \( F_1 \). This is also equivalent to convex ordering, denoted by \( F_1 \triangleleft_{c} F_2 \). If the failure rates exist, an equivalent formulation is that
\[
r_1(F_1^{-1}(z)) - r_2(F_2^{-1}(z)) \quad \text{is nondecreasing in } z \in [0, 1].
\]

**Definition 5.2.** A function \( f(t) \) defined on \([0, \infty)\) such that \((1/t)f(t)\) is increasing on \([0, \infty)\) is called star-shaped. \( F_1 \) is more IFR than \( F_2 \) (written \( F_1 <_{\text{IFR}} F_2 \)) if \( F_2^{-1}(F_1(t)) \) is star-shaped. This is also equivalent to star-ordering, denoted by \( F_1 \triangleleft_{s} F_2 \).

**Definition 5.3.** A function \( \Psi \) is superadditive if \( \Psi(x + y) \geq \Psi(x) + \Psi(y) \). A distribution function \( F_1 \) is superadditive with respect to \( F_2 \) if
\[
F_2^{-1}(F_1(x + y)) \geq F_2^{-1}(F_1(x)) + F_2^{-1}(F_1(y))
\]
for all \( x \) and \( y \) in the support of \( F_1 \).

**Definition 5.4.** \( F_1 \) is more NBU than \( F_2 \) (written \( F_1 <_{\text{NBU}} F_2 \)) if \( F_1 \) is superadditive with respect to \( F_2 \); i.e., \( F_2^{-1}(F_1(t)) \) is superadditive. This is also equivalent to superadditive ordering, denoted by \( F_1 <_{\text{SU}} F_2 \).

Each of the above orderings is scale invariant, and has the property that for a standard exponential distribution \( F_2(t) = 1 - e^{-t} \), if \( F_1 \) has aging property \( \kappa \), then \( F_1 <_{\kappa} F_2 \) for \( \kappa \in \{\text{IFR}, \text{IFRA}, \text{NBU}\} \). It also follows that \( F_1 <_{\text{IFR}} F_2 \Rightarrow F_1 <_{\text{IFRA}} F_2 \Rightarrow F_1 <_{\text{NBU}} F_2 \).

**Theorem 5.1.** Let \( G_{X_1} \) and \( G_{X_2} \) be distribution functions of random effects \( X_1 \) and \( X_2 \) for corresponding degradation paths \( \varphi_1 \) and \( \varphi_2 \) in a multiplicative degradation model, then

\[ \begin{align*}
& (i) \text{ If } G_{X_1} <_{\text{IFR}} G_{X_2} \text{ and } \eta(t) \text{ is a decreasing function of } t, \text{ then } F_{MD_1} <_{\text{IFR}} F_{MD_2}. \\
& (ii) \text{ If } G_{X_1} <_{\text{NBU}} G_{X_2} \text{ and } \eta(t) \text{ is a convex, increasing function of } t, \text{ then } F_{MD_1} <_{\text{NBU}} F_{MD_2}.
\end{align*} \]

**Proof.** (i) For the distribution functions of the DDPS \( \varphi_1 \) and \( \varphi_2 \), \( F_{MD_1}(t) = G_{X_1}(\varphi_1(\eta(t))), \quad i = 1, 2. \)
Let \( Z(t) = \zeta(t) \). Then

\[
F_{MD,2}^{-1}(F_{MD,1}(t)) = \{ G_X (Z(t)) \}^{-1}
\]

\[
= Z^{-1}\{ G_X (Z(t)) \}.
\]

Since \( G_X^{-1}(G_X(x)) \) is a convex (star-shaped) function, \( G_X^{-1}(G_X(Z(t))) \) is also a convex (star-shaped) function for an increasing function \( Z(t) \) with respect to \( t \), \( t \geq 0 \). For any monotonic increasing function \( Z(t) \), the inverse function \( Z^{-1}(t) \) is also increasing. Consequently, an increasing function of a convex (star-shaped) function, \( Z^{-1}\{ G_X^{-1}(G_X(Z(t))) \} \) is also a convex (star-shaped) function, which proves that \( F_{MD,1} < \text{IFR} F_{MD,2} \).

(ii) Since \( \eta(t) \) is convex, \( Z(t) = \varTheta t/\eta(t) \) and \( Z^{-1}(t) \) are concave, hence \( Z(t_1 + t_2) \geq Z(t_1) + Z(t_2) \) for arbitrary values of \( t_1 \geq 0 \) and \( t_2 \geq 0 \). Therefore,

\[
Z^{-1}\{ G_X^{-1}(G_X(Z(t_1 + t_2))) \}
\]

\[
\geq Z^{-1}\{ G_X^{-1}(G_X(Z(t_1)) + G_X(Z(t_2))) \}
\]

\[
\geq Z^{-1}\{ G_X^{-1}(G_X(Z(t_1))) \} + Z^{-1}\{ G_X^{-1}(G_X(Z(t_2))) \},
\]

which proves \( F_{MD,1} < \text{NBU} F_{MD,2} \). □

6. Examples

In this section we provide some illustrations of degradation path models that lead to specific lifetime distributions. Beyond the scope in Lu and Meeker [5], lifetime distributions with BT shaped failure rate are derived using the relationship between \( G_X \) and \( \eta(t) \).

6.1. \( X \) is Weibull-distributed

Consider the simple IDP \( \vartheta(t; X, \Theta) = \zeta(t) \), where \( X = \zeta(t) > 0 \) is the degradation rate. Assuming \( X \) is Weibull distributed with cdf

\[
G_X(x) = 1 - \exp[-(x/\lambda)^\kappa], \quad \lambda, \kappa > 0
\]

then the lifetime distribution is

\[
F_M(t) = \Pr[\zeta(t) \geq \varTheta t] = \Pr\left[ \zeta(t) \geq \frac{\varTheta t}{t} = \exp\left[ -\left(\frac{\varTheta t}{t} \right)^\kappa \right] \right]
\]

for \( \varTheta t \geq 0 \), and by letting \( v = \lambda \varTheta t \),

\[
F_M(t) = \exp[-(v/t)^\kappa].
\]

This is a cdf of the rarely applied inverse-Weibull distribution. Huang and Askin [10] applied the Weibull distribution to characterize item-to-item variability of electronic devices which degrade linearly, but the inverse-Weibull lifetime distribution (and its difficulty in use) is not discussed. The failure rate of inverse-Weibull distribution is given by

\[
r_M(t) = \frac{\kappa t^{\kappa-1} \exp[-(v/t)^\kappa]}{1 - \exp[-(v/t)^\kappa]},
\]

which has a unimodal failure rate with \( \lim_{t \to 0} r_M(t) = \lim_{t \to \infty} r_M(t) = 0 \), in strong contrast to the Weibull distribution.

For the DDP, we consider the model used by Fukada [11] to characterize the degradation of electronic devices:

\[
\vartheta(t; X, \Theta) = \theta_3 \cdot \exp[-\exp((\theta_1 t)^{\theta_2})], \quad \theta_1, \theta_2 > 0, \quad \theta_3 \geq 1.
\]

(23)

After a logarithmic transformation,

\[
\log \vartheta(t; X, \Theta) = \log \vartheta_3 - \exp((\theta_1 t)^{\theta_2}).
\]

If \( X \equiv \log \theta_3 (\geq 0) \) is assumed to be a random effect that follows a Weibull distribution with cdf given in (21), then for \( \eta(t) = -\exp((\theta_1 t)^{\theta_2}) \), which satisfies \( \varTheta t - \eta(t) \geq 0 \), we obtain the lifetime distribution

\[
F_{AD}(t) = G_X(\varTheta t - \eta(t))
\]

\[
= 1 - \exp[-\lambda(\varTheta t + \exp((\theta_1 t)^{\theta_2}))^\kappa].
\]

In the case \( \kappa = \lambda = 1 \), i.e., \( X \) has a standard exponential distribution, the survivor function is given by

\[
\tilde{F}_{AD}(t) = \exp[-\varTheta t - \exp((\theta_1 t)^{\theta_2})].
\]

(24)

Suppose that a failure is considered at the threshold value \( \varTheta t = \log \vartheta_3 = -1 \). Then (24) is reduced to the survivor function of an exponential power distribution with a failure rate \( r_{AD}(t) = \theta_3 \theta_2 (\theta_1 t)^{\theta_2 - 1} \exp[(\theta_1 t)^{\theta_2}] \). The exponential power distribution is one of the few tractable two-parameter distributions that possesses a BT shaped failure rate (see Fig. 3). Its failure rate is BT shaped when \( \theta_2 < 1 \), achieving a minimum at \( [(1 - \theta_2)/(\theta_1 \theta_2)]^{1/\theta_2} \). For \( \theta_2 = 1 \), the exponential power distribution reduces to an extreme value distribution [12].

Next, consider the following DDP where a random effect is entered into the model multiplicatively:

\[
\vartheta(t; X, \Theta) = \theta_3 \cdot [\log(\theta_1 t + 1)]^{-\theta_2}, \quad \theta_1 > 0, \quad i = 1, 2, 3
\]

(25)

with deterministic degradation function \( [\log(\theta_1 t + 1)]^{-\theta_2} \). Suppose that \( X \equiv \theta_3 \) follows a Weibull distribution with cdf given by (21). Then, combining (9) and (21), a lifetime distribution function derived from degradation

\[
\text{Fig. 3. Failure rate plot of exponential power distribution.}
\]
model (25) is
\[ F_{MD}(t) = 1 - \exp[-(\lambda D_t)^{\gamma} \cdot \log(\theta_1 t + 1)]^{\alpha_2}. \] (26)
In the case \( \kappa = 1, \lambda D_t = 1, \) and \( \theta_2 \geq 1, \) distribution (26) can be expressed as
\[ F_{MD}(t) = 1 - \exp[-(\log(\theta_1 t + 1))^{\beta_2 + 1}], \quad \beta_2 \geq 0. \]
This is called a two-parameter distribution II in [12], and its failure rate
\[ r_{MD}(t) = \frac{\theta_2 (\beta_2 + 1) \cdot (\log(\theta_1 t + 1))^{\beta_2}}{\theta_1 t + 1} \]
can exhibit both an increasing and decreasing failure pattern, as shown in Fig. 4.

6.2. \( X \) is gamma-distributed

Suppose that the random effect \( X \) follows gamma distribution with a scale parameter \( \lambda > 0 \) and a shape parameter \( \gamma > 0. \) The pdf of \( X \) is
\[ g_X(x) = \frac{\lambda^\gamma x^{\gamma-1} e^{-\lambda x}}{\Gamma(\gamma)}, \quad x \geq 0, \]
where \( \Gamma(\cdot) \) denotes the well-known gamma function.

To investigate general characteristics of the lifetime distribution from a gamma-distributed degradation path, for example, we consider the following metal corrosion process. The rate of metal corrosion decreases in time as 
\[ \frac{d\eta(t)}{dt} = \theta_2 / (\theta_1 t + 1), \]
where \( \theta_1 \) and \( \theta_2 > 0 \) are material-specific constants, hence the mean degradation path is
\[ \mathcal{D}(t; X, \Theta) = \theta_2 \log(\theta_1 t + 1) \] [13]. When we assume \( X = \theta_2 \) is gamma-distributed, the failure-time distribution is
\[ F_{MD}(t) = 1 - G_X \left( \frac{\mathcal{D}_t}{\eta(t)} \right) = 1 - \int_0^{\mathcal{D}_t / \eta(t)} \frac{\lambda^\gamma x^{\gamma-1} e^{-\lambda x}}{\Gamma(\gamma)} \, dx \]
\[ = 1 - \int_0^{\mathcal{D}_t / (\log(\theta_1 t + 1))} \frac{\lambda^\gamma x^{\gamma-1} e^{-\lambda x}}{\Gamma(\gamma)} \, dx, \] (27)
and the pdf is obtained by differentiating (27)
\[ f_{MD}(t) = \frac{(\lambda D_t)^\gamma e^{-(\lambda D_t) \log(\theta_1 t + 1)}}{\Gamma(\gamma) (\log(\theta_1 t + 1))^{\gamma+1}}, \quad t \geq 0. \] (28)
Its failure rate can be obtained by using (27) and (28). While the gamma distribution’s failure rate can be increasing \( (\gamma > 1), \) constant \( (\gamma = 1) \) or decreasing \( (\gamma < 1), \) failure rates of the metal corrosion process with a gamma distributed item-to-item variability are always unimodal regardless of the shape parameter’s value as shown in Fig. 5. In other words, despite the apparent flexibility of the degradation model, we will be stuck with a UBT failure rate in the lifetime distribution, whether or not it makes intuitive sense in the experiment.

6.3. \( X \) is log-logistically distributed

The pdf of a log-logistic or a Weibull-exponential distribution [14] is
\[ g_X(x) = \beta x^{\beta-1} e^{-x^\beta} / (1 + e^{x^\beta})^2, \quad x > 0, \quad \beta > 0, \]
with cdf
\[ G_X(x) = 1 - \frac{1}{1 + e^{x^\beta}}. \]
The log-logistic distribution is in the Burr type-XII family of distributions. Consider the following time-varying decay model of the ultraviolet (UV)-induced fiber Bragg gratings in Erdogan et al. [15]:
\[ \mathcal{D}(t; X, \Theta) = \theta_3 (\theta_1 t + 1)^{\beta_2}, \quad \theta_1 > 0, \quad i = 1, 2, 3, \]
with a random effect \( X = \theta_3. \) If \( X \) follows a log-logistic distribution with parameters \( z \) and \( \beta, \) the lifetime distribution of \( \mathcal{D} \) is
\[ F_{MD}(t) = 1 - \frac{1}{1 + e^{z [\mathcal{D}_t (\theta_1 t + 1)^{\beta_2}]^\beta}}, \]
and by letting $b = e^{-\frac{2}{\theta_2}} \left( \frac{t}{\theta_1} \right)^{-1}$, 
\[ F_{MD}(t) = 1 - \left[ 1 + \left( \frac{t}{\theta_1} \right)^{-\theta_2} \right]^{-1}, \quad b > 0, \ t \geq 0. \quad (29) \]

In the special case where $\theta_2 = 2$, distribution (29) represents $b(F_{2,2}^{(1,1)})$, where $F_{2,2}$ denotes a (central) $F$ random variable with (2,2) degrees of freedom. The pdf of $Z = bF_{2,2}^{1/2}$ is 
\[ f_Z(z) = \frac{(b/2)^{-1}(z/b)}{B(1,1)(1 + (z/b)^2)} \frac{1}{2}, \]
where $B(\cdot)$ denotes the beta function. The resulting failure rate 
\[ r_{MD}(t) = \frac{2t}{b^2 + t^2} \]
increases up until time $t = b$ and then decreases (see Fig. 6).

7. Conclusion

In modeling material fatigue and degradation, the relationship between the randomness in degradation and randomness in the resulting lifetime distribution is strong and direct, albeit hard to discern. By making tacit assumptions about the degradation distribution, the resulting implications to the lifetime distribution may surprise the experimenter and possibly contradict the assumptions about the failure characteristics in the study.

This research reveals surprising restrictions on the properties of the failure time distribution when assumptions are made about the degradation distribution. Interestingly, some degradation models, under distributional assumption of random parameters in the model, directly imply lifetime distributions having BT-shaped failure rates.

The comparisons in this article are based on basic degradation functions that are applied to show the relationships between degradation and lifetime with minimal complexity. Certainly, most practical applications would use degradation functions extended beyond the elementary ones listed here. However, these results bolster the need for further attention to this implied relationship of interest; few efforts have been pursued toward this end.

While degradation modeling is growing beyond the purview of manufacturing and materials testing (e.g., logistics-performance degradation in supply-chain networks [16]), reliability prediction through degradation modeling will be further emphasized as a supporting tool in lifetime data analysis. Even if the degradation model (including the uncertainty) can be derived from sound physical principles from the experiment, we suggest that the lifetime distribution should be carefully studied to ensure it also reflects the assumptions of the experimenter.

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